

The Einstein-Hilbert action of the space of holomorphic maps from S^2 to $\mathbb{C}P^k$

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Abstract

Let $\mathcal{H}_{n,k}(\Sigma)$ be the space of degree $n \geq 1$ holomorphic maps from a compact Riemann surface Σ to $\mathbb{C}P^k$. In the case $\Sigma = S^2$ and $n = 1$, the L^2 metric on $\mathcal{H}_{1,k}(S^2)$ was computed exactly by Speight. In this paper, the Ricci curvature tensor and the scalar curvature on $\mathcal{H}_{1,k}(S^2)$ are determined explicitly for $k \geq 2$. An exact direct computation of the Einstein-Hilbert action with respect to the L^2 metric on $\mathcal{H}_{1,k}(S^2)$ is made and shown to coincide with a formula conjectured by Baptista.

1 Introduction

Let Σ be a compact Riemann surface equipped with a Riemannian metric g and let h be the Fubini-Study metric on $\mathbb{C}P^k$. Let ϕ be a holomorphic map from Σ to $\mathbb{C}P^k$ of degree $n \geq 1$ defined as

$$n = \int_{\Sigma} \phi^* \omega_0, \quad (1)$$

where ω_0 is the normalized Kähler form with respect to h . Consider the space of degree n holomorphic maps $\Sigma \rightarrow \mathbb{C}P^k$, denoted $\mathcal{H}_{n,k}(\Sigma)$. There is a natural Riemannian metric on $\mathcal{H}_{n,k}(\Sigma)$ defined by the metrics g and h on Σ and $\mathbb{C}P^k$ as

$$\gamma_{L^2}(X, Y) = \int_{\Sigma} h(X, Y) \text{vol}_g, \quad (2)$$

for $X, Y \in T_{\phi} \mathcal{H}_{n,k}(\Sigma) \subset \Gamma(\phi^* T\mathbb{C}P^k)$. This is called the L^2 metric on $\mathcal{H}_{n,k}(\Sigma)$.

In the physics literature, the degree n holomorphic map ϕ is regarded as a $\mathbb{C}P^k$ lump of charge n on Σ , that is, a degree n minimal energy static solution of the field equations of the $\mathbb{C}P^k$ model on Σ . Hence, the degree n moduli space \mathcal{M}_n of the $\mathbb{C}P^k$ model on Σ is $\mathcal{H}_{n,k}(\Sigma)$. The low energy dynamics of $\mathbb{C}P^k$ lumps is conjecturally approximated by geodesic motion

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on \mathcal{M}_n with respect to the L^2 metric γ_{L^2} [5, 7, 10]. A precise version of this conjecture is proved for $\Sigma = T^2$ and $n \geq 2$ by Speight in [9].

With respect to the L^2 metric, Baptista [1] has given conjectural formulae for the volume and the Einstein-Hilbert action of $\mathcal{H}_{n,k}(\Sigma)$, provided Σ has genus $g \leq n/2$,

$$\text{Vol}(\mathcal{H}_{n,k}(\Sigma), \gamma_{L^2}) = \frac{(k+1)^g}{m!} \left(\pi \text{Vol}(\Sigma, g) \right)^m, \quad (3)$$

$$H(\mathcal{H}_{n,k}(\Sigma), \gamma_{L^2}) = \frac{2\pi(k+1)^g [m - 2g + 1]}{(m-1)!} \left(\pi \text{Vol}(\Sigma, g) \right)^{m-1}, \quad (4)$$

where $m = (k+1)(n+1-g) + g - 1$ and $\text{Vol}(\Sigma, g)$ is the volume of Σ . This conjecture is based on a singular limit relating the $\mathbb{C}P^k$ model on Σ with a gauged sigma model whose fields take values in \mathbb{C}^{k+1} [1]. More precisely, a one parameter family of metrics on the n -vortex moduli space, which is a compact Kähler manifold, are conjectured to converge, in a certain limit, to the L^2 metric on $\mathcal{H}_{n,k}(\Sigma)$. Such convergence has recently been established rigorously by Lui [6] in the sense of Cheeger-Gromov, that is on each open set is some locally finite open cover of $\mathcal{H}_{n,k}(\Sigma)$. This convergence does not directly imply Baptista's conjectured formulae for the volume and the Einstein-Hilbert action of $\mathcal{H}_{n,k}(\Sigma)$, however.

In the case $n = 1$ and $\Sigma = S^2$, Speight [7, 8] has determined an explicit formula for the L^2 metric on $\mathcal{H}_{1,k}(S^2)$, and then computed the volume of $\mathcal{H}_{1,k}(S^2)$ for $k \geq 2$ finding agreement with the conjectural formula (3). In this paper, an explicit formula for the Ricci curvature tensor, and then the scalar curvature on $(\mathcal{H}_{1,k}(S^2), \gamma_{L^2})$ have been determined for $k \geq 2$, by exploiting the Kähler property of the L^2 metric. The Einstein-Hilbert action of $\mathcal{H}_{1,k}(S^2)$ with respect to the L^2 metric is computed for $k \geq 2$ confirming the formula (4).

2 Degree 1 Holomorphic Maps $S^2 \rightarrow \mathbb{C}P^k$

This section reviews the geometric structure of $\mathcal{H}_{1,k}(S^2)$ introduced in [8]. Let S^2 be the 2-sphere equipped with the standard round metric and let ϕ be a degree 1 holomorphic map $S^2 \rightarrow \mathbb{C}P^k$. Introducing homogeneous coordinates (z_0, z_1) on $\mathbb{C}P^1 \cong S^2$, then such degree 1 map has the form

$$\phi([z_0, z_1]) = [a_0 z_0 + b_0 z_1, \dots, a_k z_0 + b_k z_1], \quad (5)$$

where (a_0, \dots, a_k) and (b_0, \dots, b_k) are linearly independent in \mathbb{C}^{k+1} . Since the elements $(\xi a_0, \xi b_0, \dots, \xi a_k, \xi b_k) \in \mathbb{C}^{2k+2}$, where $\xi \in \mathbb{C}^\times$, determine the same holomorphic map ϕ , then there is an open inclusion $\mathcal{H}_{1,k}(S^2) \hookrightarrow \mathbb{C}P^{2k+1}$ which is used to equip $\mathcal{H}_{1,k}(S^2)$ with a topology, differentiable and complex structures.

The isometry groups $U(2)$ and $U(k+1)$ of $\mathbb{C}P^1$ and $\mathbb{C}P^k$ respectively build an isometric action of $G = U(k+1) \times U(2)$ on $\mathcal{H}_{1,k}(S^2)$, that is, $\phi \rightarrow \sigma_2 \circ \phi \circ \sigma_1^{-1}$ where σ_1 and σ_2 are

isometries of $\mathbb{C}P^1$ and $\mathbb{C}P^k$. Generically, each orbit of G on $\mathcal{H}_{1,k}(S^2)$ is a real codimension 1 submanifold of $\mathcal{H}_{1,k}(S^2)$ and has a unique element ϕ_μ given by

$$\phi_\mu([z_0, z_1]) = [\mu z_0, z_1, 0, \dots, 0], \quad \mu > 1. \quad (6)$$

An exceptional orbit of real codimension 3 occurs when $\mu = 1$. This action decomposes $\mathcal{H}_{1,k}(S^2)$ into a one parameter family of orbits parametrized by $\mu \in [1, \infty)$. For $\mu > 1$, the isotropy group of the orbit G_μ of ϕ_μ is

$$K = \left\{ \left(\begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & U \end{pmatrix}, \begin{pmatrix} e^{i(\alpha+\delta)} & 0 \\ 0 & e^{i(\beta+\delta)} \end{pmatrix} \right) : \alpha, \beta, \delta \in \mathbb{R}, U \in U(k-1) \right\}. \quad (7)$$

By the Orbit-Stabilizer Theorem, each orbit G_μ is diffeomorphic to G/K .

Now, let \mathfrak{g} and \mathfrak{k} denote the Lie algebras of G and K respectively and \langle, \rangle be the $Ad(G)$ invariant inner product on \mathfrak{g} ,

$$\langle (M_1, m_1), (M_2, m_2) \rangle = -\frac{1}{2}(\text{tr} M_1 M_2 + \text{tr} m_1 m_2), \quad (8)$$

where $M_i \in \mathfrak{u}(k+1)$ and $m_i \in \mathfrak{u}(2)$. The tangent space of $\mathcal{H}_{1,k}(S^2)$ at ϕ_μ is

$$V_\mu := T_{\phi_\mu} \mathcal{H}_{1,k}(S^2) = \left\langle \frac{\partial}{\partial \mu} \right\rangle \oplus \mathfrak{p}, \quad (9)$$

where \mathfrak{p} is the orthogonal complement of \mathfrak{k} in \mathfrak{g} with respect to \langle, \rangle . The space \mathfrak{p} can be decomposed into $Ad(K)$ invariant subspaces

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_\mu \oplus \tilde{\mathfrak{p}}_\mu \oplus \hat{\mathfrak{p}} \oplus \check{\mathfrak{p}}, \quad (10)$$

where \mathfrak{p}_0 is a 1 real-dimensional space, $\mathfrak{p}_\mu, \tilde{\mathfrak{p}}_\mu$ are 1-complex dimensional subspaces depending on μ , and $\hat{\mathfrak{p}}$ and $\check{\mathfrak{p}}$ are $(k-1)$ complex dimensional subspaces. The definitions of these subspaces are included in the Appendix. It was shown in [8] that

Proposition 1. *Let γ be a G invariant Kähler metric on $\mathcal{H}_{1,k}(S^2)$. Then, for $k \geq 2$, γ is uniquely determined by the one parameter family of symmetric bilinear forms $\gamma_\mu : V_\mu \times V_\mu \rightarrow \mathbb{R}$ given by*

$$\gamma_\mu = A_0(\mu) d\mu^2 + 8\mu^2 A_0(\mu) \langle, \rangle_{\mathfrak{p}_0} + A_1(\mu) \langle, \rangle_{\mathfrak{p}_\mu} + A_2(\mu) \langle, \rangle_{\tilde{\mathfrak{p}}_\mu} + A_3(\mu) \langle, \rangle_{\hat{\mathfrak{p}}} + A_4(\mu) \langle, \rangle_{\check{\mathfrak{p}}}, \quad (11)$$

where A_0, \dots, A_4 are smooth positive functions of μ defined by a single function $A(\mu)$ and a positive constant B as follows

$$\begin{aligned} A_0(\mu) &= \frac{1}{4\mu} A'(\mu), & A_1(\mu) &= A_2(\mu) = \frac{\mu^2 - 1}{\mu^2 + 1} A(\mu), \\ A_3(\mu) &= B + \frac{A(\mu)}{2}, & A_4(\mu) &= B - \frac{A(\mu)}{2}, \end{aligned} \quad (12)$$

and $\langle, \rangle_{\mathfrak{p}_i}$ denote the induced inner products of \langle, \rangle on the $Ad(K)$ invariant subspaces, given in (10).

For the L^2 metric γ_{L^2} on $\mathcal{H}_{1,k}(S^2)$, the function $A(\mu)$ and the constant B are

$$A_{L^2}(\mu) = \frac{16\pi}{c_1 c_2} \frac{\mu^4 - 4\mu^2 \log \mu - 1}{(\mu^2 - 1)^2}, \quad B_{L^2} = \frac{8\pi}{c_1 c_2}, \quad (13)$$

where c_1 and c_2 are the constant holomorphic sectional curvatures of g and h respectively. Another G invariant Kähler metric on $\mathcal{H}_{1,k}(S^2)$ is the induced metric defined by the inclusion $\mathcal{H}_{1,k}(S^2) \hookrightarrow \mathbb{C}P^{2k+1}$, where $\mathbb{C}P^{2k+1}$ is given the Fubini-Study metric (of constant holomorphic sectional curvature c , say). We call this the Fubini-Study metric on $\mathcal{H}_{1,k}(S^2)$, denoted γ_{FS} . It is determined by

$$A_{FS}(\mu) = \frac{4}{c} \frac{\mu^2 - 1}{\mu^2 + 1}, \quad B_{FS} = \frac{2}{c}. \quad (14)$$

The volume form of a G invariant Kähler metric γ , determined as in (11) by the function $A(\mu)$ and the constant B , on $\mathcal{H}_{1,k}(S^2)$ is

$$\text{vol}_\gamma = V(\mu) d\mu \wedge \text{vol}_{G/K}, \quad (15)$$

where

$$V(\mu) = \frac{1}{\sqrt{2}} A^2 \left(B^2 - \frac{A^2}{4} \right)^{k-1} A'(\mu), \quad (16)$$

and $\text{vol}_{G/K}$ is the volume form of G/K with respect to the inner product \langle, \rangle , defined in (8). It was shown that for $k \geq 2$, every G invariant Kähler metric γ on $\mathcal{H}_{1,k}(S^2)$ has finite volume[8]. In fact, if $\lim_{\mu \rightarrow \infty} A(\mu) = 2B$, this volume is

$$\text{Vol}(\mathcal{H}_{1,k}(S^2), \gamma) = \sqrt{2} (2B)^{2k+1} \frac{(k-1)!k!}{(2k+1)!} \text{Vol}(G/K) = \frac{(2B\pi)^{2k+1}}{(2k+1)!}, \quad (17)$$

where $\text{Vol}(G/K)$ is the volume of G/K with respect to \langle, \rangle .

3 Ricci Curvature Tensor

With respect to any G invariant Kähler metric γ , determined as in Proposition 1, on $\mathcal{H}_{1,k}(S^2)$, we determine an explicit formula for the Ricci curvature tensor ρ as follows

Proposition 2. *Let γ be a G invariant Kähler metric on $\mathcal{H}_{1,k}(S^2)$, determined as in (11) by the function $A(\mu)$ and the constant B . Then, the Ricci curvature tensor ρ on $(\mathcal{H}_{1,k}(S^2), \gamma)$ with $k \geq 2$ is uniquely determined by the one parameter family of symmetric bilinear forms $\rho_\mu : V_\mu \times V_\mu \rightarrow \mathbb{R}$, given by*

$$\rho_\mu = C_0 d\mu^2 + 8\mu^2 C_0 \langle, \rangle_{\mathfrak{p}_0} + C_1(\mu) \langle, \rangle_{\mathfrak{p}_\mu} + C_2(\mu) \langle, \rangle_{\tilde{\mathfrak{p}}_\mu} + C_3(\mu) \langle, \rangle_{\tilde{\mathfrak{p}}} + C_4(\mu) \langle, \rangle_{\tilde{\mathfrak{p}}}, \quad (18)$$

where C_0, \dots, C_4 are smooth functions of μ , determined as in (12), by the function $C(\mu)$ and the constant D given by

$$C(\mu) = 4(k+1) \frac{\mu^2 - 1}{\mu^2 + 1} - 2\mu \frac{F'(\mu)}{F(\mu)}, \quad D = 2(k+1), \quad (19)$$

where

$$F(\mu) = \frac{A^2 A'(\mu)}{A_{FS}^2 A'_{FS}(\mu)} \left(B^2 - \frac{A^2}{4} \right)^{k-1} \left(B_{FS}^2 - \frac{A_{FS}^2}{4} \right)^{-(k-1)}. \quad (20)$$

Proof: The Ricci curvature tensor ρ on $(\mathcal{H}_{1,k}(S^2), \gamma)$ is a G invariant symmetric $(0, 2)$ tensor which is Hermitian and its associated 2-form $\hat{\rho} = \rho(J, \cdot)$ is closed. Hence, ρ has the same structure as γ , that is, it is uniquely determined by the one parameter family of symmetric bilinear forms $\rho_\mu : V_\mu \times V_\mu \rightarrow \mathbb{R}$, given as in (11). Since the coefficients C_0, \dots, C_4 are defined as in (12) by a single function $C(\mu)$ and a constant D , then we only need to determine $C(\mu)$ and D . By Proposition 1, we have

$$C(\mu) = C_3(\mu) - C_4(\mu), \quad D = \frac{1}{2}[C_3(\mu) + C_4(\mu)]. \quad (21)$$

To compute $C(\mu)$ and D , we need first an orthonormal basis for \mathfrak{p} with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$. We shall use the orthonormal basis $\{Y_i, \hat{Y}_j, \check{Y}_j : i = 0, \dots, 4, j = 1, \dots, 2k-2\}$ introduced in [8]. The structure of this basis is included in the Appendix. Hence, the functions C_3 and C_4 can be given, for example, by

$$\begin{aligned} C_3 &= \rho_\mu(\hat{Y}_1, \hat{Y}_1) = -\rho_\mu(J\hat{Y}_2, \hat{Y}_1) = \hat{\rho}_\mu(\hat{Y}_1, \hat{Y}_2), \\ C_4 &= \rho_\mu(\check{Y}_1, \check{Y}_1) = -\rho_\mu(J\check{Y}_2, \check{Y}_1) = \hat{\rho}_\mu(\check{Y}_1, \check{Y}_2). \end{aligned} \quad (22)$$

Now, the volume form, given in (15), of any G invariant Kähler metric γ on $\mathcal{H}_{1,k}(S^2)$ can be written as

$$\text{vol}_\gamma = F(\mu) \text{vol}_{\gamma_{FS}}, \quad (23)$$

where

$$F(\mu) = \frac{A^2 A'(\mu)}{A_{FS}^2 A'_{FS}(\mu)} \left(B^2 - \frac{A^2}{4} \right)^{k-1} \left(B_{FS}^2 - \frac{A_{FS}^2}{4} \right)^{-(k-1)}. \quad (24)$$

Hence, the Ricci form $\hat{\rho}$ with respect to γ is [2],

$$\hat{\rho} = \hat{\rho}_{FS} - i\partial\bar{\partial}f, \quad f(\mu) := \log F(\mu), \quad (25)$$

where $\hat{\rho}_{FS}$ is the Ricci form with respect to γ_{FS} , $\partial : \Omega^{(p,q)} \rightarrow \Omega^{(p+1,q)}$, and $\bar{\partial} : \Omega^{(p,q)} \rightarrow \Omega^{(p,q+1)}$ are the partial exterior derivatives on the space of (p, q) -forms $\Omega^{(p,q)}$ on $\mathcal{H}_{1,k}(S^2)$. Using (25) in (22), we have

$$\begin{aligned}
C(\mu) &= \hat{\rho}_{\mu FS}(\hat{Y}_1, \hat{Y}_2) - \hat{\rho}_{\mu FS}(\check{Y}_1, \check{Y}_2) - i[(\partial\bar{\partial}f)_\mu(\hat{Y}_1, \hat{Y}_2) - (\partial\bar{\partial}f)_\mu(\check{Y}_1, \check{Y}_2)], \\
&= C_{FS}(\mu) - i[(\partial\bar{\partial}f)_\mu(\hat{Y}_1, \hat{Y}_2) - (\partial\bar{\partial}f)_\mu(\check{Y}_1, \check{Y}_2)],
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
2D &= \hat{\rho}_{\mu FS}(\hat{Y}_1, \hat{Y}_2) + \hat{\rho}_{\mu FS}(\check{Y}_1, \check{Y}_2) - i[(\partial\bar{\partial}f)(\mu)(\hat{Y}_1, \hat{Y}_2) + (\partial\bar{\partial}f)(\mu)(\check{Y}_1, \check{Y}_2)], \\
&= 2D_{FS} - i[(\partial\bar{\partial}f)_\mu(\hat{Y}_1, \hat{Y}_2) + (\partial\bar{\partial}f)_\mu(\check{Y}_1, \check{Y}_2)].
\end{aligned} \tag{27}$$

Since $(\mathcal{H}_{1,k}(S^2), \gamma_{FS})$ is a $(2k+1)$ complex dimensional Kähler-Einstein manifold, then [4]

$$\hat{\rho}_{FS} = c(k+1)\omega_{FS}, \tag{28}$$

where ω_{FS} is the Kähler form of γ_{FS} . Hence, the function $C_{FS}(\mu)$ and the constant D_{FS} are

$$C_{FS}(\mu) = c(k+1)A_{FS} = 4(k+1)\frac{\mu^2 - 1}{\mu^2 + 1}, \quad D_{FS} = c(k+1)B_{FS} = 2(k+1). \tag{29}$$

It remains to compute $(\partial\bar{\partial}f)_\mu(\hat{Y}_1, \hat{Y}_2)$ and $(\partial\bar{\partial}f)_\mu(\check{Y}_1, \check{Y}_2)$. Let $\xi_0 = -Y_0/(2\sqrt{2}\mu)$, then the Hermiticity of γ implies that $J\xi_0 = -\partial/\partial\mu$, and so,

$$(J^*d\mu)(\xi_0) = d\mu(J\xi_0) = d\mu(-\frac{\partial}{\partial\mu}) = -1, \tag{30}$$

where J^* is the induced almost complex structure on V_μ^* . This means that $\eta_0 = -J^*d\mu$ is the covector of ξ_0 , that is, $\eta_0(\xi_0) = 1$. The exterior derivative of f is

$$df = \frac{1}{2}f'(\mu)[(d\mu + i\eta_0) + (d\mu - i\eta_0)] = \frac{1}{2}f'(\mu)[(d\mu - iJ^*d\mu) + (d\mu + iJ^*d\mu)]. \tag{31}$$

This implies that the $(1,0)$ -part ∂f and the $(0,1)$ -part $\bar{\partial}f$ of the 1-form df are

$$\partial f = \frac{1}{2}f'(\mu)(d\mu + i\eta_0), \quad \bar{\partial}f = \frac{1}{2}f'(\mu)(d\mu - i\eta_0). \tag{32}$$

Since $d = \partial + \bar{\partial}$ and $\bar{\partial}^2 = 0$, then

$$\partial\bar{\partial}f = d\bar{\partial}f = -\frac{i}{2}f''(\mu)d\mu \wedge \eta_0 - \frac{i}{2}f'(\mu)d\eta_0, \tag{33}$$

where $d\eta_0$ is a 2-form on $\mathcal{H}_{1,k}(S^2)$ given for any vector fields X, Y on $\mathcal{H}_{1,k}(S^2)$ by

$$d\eta_0(X, Y) = X[\eta_0(Y)] - Y[\eta_0(X)] - \eta_0([X, Y]). \tag{34}$$

Let ξ_1, ξ_2 be the extension of \hat{Y}_1 and \hat{Y}_2 as Killing vector fields on $\mathcal{H}_{1,k}(S^2)$. Then, from (33) and (34), we have

$$(\partial\bar{\partial}f)_\mu(\hat{Y}_1, \hat{Y}_2) = \frac{i}{2}f'(\mu) \eta_0([\xi_1, \xi_2]\Big|_{\phi=\phi_\mu}). \quad (35)$$

The Lie bracket of Killing vector fields on $\mathcal{H}_{1,k}(S^2)$ can be defined by the Lie algebra bracket $[\cdot, \cdot]_{\mathfrak{g}}$ of \mathfrak{g} as follows [3]

$$[\xi_1, \xi_2]\Big|_{\phi=\phi_\mu} = -P_{\mathfrak{p}}([\hat{Y}_1, \hat{Y}_2]_{\mathfrak{g}}), \quad (36)$$

where $P_{\mathfrak{p}}$ is the projection of \mathfrak{g} to \mathfrak{p} . Since

$$\hat{Y}_1 = (-E_{13} + E_{31}, \mathbf{0}), \quad \hat{Y}_2 = i(E_{13} + E_{31}, \mathbf{0}), \quad (37)$$

as in the Appendix. Then, we have

$$\begin{aligned} [\hat{Y}_1, \hat{Y}_2]_{\mathfrak{g}} &= -2i(E_{13}E_{31} - E_{31}E_{13}, \mathbf{0}), \\ &= -i(2E_{11} - 2E_{33}, \mathbf{0}), \\ &= -\frac{i}{2}(3E_{11} + E_{22} - 2E_{33}, e_{11} - e_{22}) + \frac{i}{2}(E_{11} - E_{22}, -e_{11} + e_{22}), \\ &= -\frac{i}{2}(3E_{11} + E_{22} - 2E_{33}, e_{11} - e_{22}) + \frac{1}{\sqrt{2}}Y_0, \end{aligned} \quad (38)$$

where $E_{\alpha\beta}$ and $e_{\alpha\beta}$ denote $(k+1) \times (k+1)$ and 2×2 matrices respectively whose element (α, β) is 1, and the others being zero. Since the element $i(3E_{11} + E_{22} - 2E_{33}, e_{11} - e_{22})/2 \in \mathfrak{k}$, then it vanishes under $P_{\mathfrak{p}}$, and so

$$[\xi_1, \xi_2]\Big|_{\phi=\phi_\mu} = -\frac{1}{\sqrt{2}}Y_0. \quad (39)$$

Substituting (39) in (35), we get

$$(\partial\bar{\partial}f)_\mu(\hat{Y}_1, \hat{Y}_2) = i\mu f'(\mu). \quad (40)$$

Similarly, one can find that

$$(\partial\bar{\partial}f)_\mu(\check{Y}_1, \check{Y}_2) = -i\mu f'(\mu). \quad (41)$$

Substituting (29), (40) and (41) in (26) and (27), we obtain the function $C(\mu)$ and the constant D as in (19). □

4 Scalar Curvature

An orthonormal basis for (V_μ, γ_μ) can be defined as follows [8],

$$\begin{aligned}
X &= \frac{1}{\sqrt{A_0}} \frac{\partial}{\partial \mu}, & X_0 &= \frac{1}{\sqrt{8\mu^2 A_0}} Y_0, \\
X_1 &= \frac{Y_1 - \mu Y_3}{\sqrt{(1 + \mu^2) A_1}}, & X_2 &= \frac{Y_2 + \mu Y_4}{\sqrt{(1 + \mu^2) A_1}}, \\
X_3 &= \frac{-\mu Y_1 + Y_3}{\sqrt{(1 + \mu^2) A_1}}, & X_4 &= \frac{\mu Y_2 + Y_4}{\sqrt{(1 + \mu^2) A_1}}, \\
\hat{X}_j &= \frac{1}{\sqrt{A_3}} \hat{Y}_j, & \check{X}_j &= \frac{1}{\sqrt{A_4}} \check{Y}_j, \quad j = 1, \dots, 2k-2.
\end{aligned} \tag{42}$$

Proposition 3. *Let γ be a G invariant Kähler metric on $\mathcal{H}_{1,k}(S^2)$, determined as in (11) by the function $A(\mu)$ and the constant B . Then, the scalar curvature of $(\mathcal{H}_{1,k}(S^2), \gamma)$ for $k \geq 2$ is*

$$\kappa(\mu) = 2 \left[2 \frac{C(\mu)}{A(\mu)} + \frac{C'(\mu)}{A'(\mu)} \right] + 2(k-1) \left[\frac{4(k+1) + C(\mu)}{2B + A(\mu)} + \frac{4(k+1) - C(\mu)}{2B - A(\mu)} \right]. \tag{43}$$

Proof: The scalar curvature of a G invariant Kähler metric γ , determined as in (11), with respect to the orthonormal basis (42) is

$$\begin{aligned}
\kappa(\mu) &= \rho_\mu(X, X) + \sum_{i=0}^4 \rho_\mu(X_i, X_i) + \sum_{j=1}^{2k-2} [\rho_\mu(\hat{X}_j, \hat{X}_j) + \rho_\mu(\check{X}_j, \check{X}_j)], \\
&= \frac{1}{A_0} \rho_\mu\left(\frac{\partial}{\partial \mu}, \frac{\partial}{\partial \mu}\right) + \frac{8\mu^2}{A_0} \rho_\mu(Y_0, Y_0) + \frac{1}{A_1} \sum_{i=1}^4 \rho_\mu(Y_i, Y_i) \\
&\quad + \frac{1}{A_3} \sum_{j=1}^{2k-2} \rho_\mu(\hat{Y}_j, \hat{Y}_j) + \frac{1}{A_4} \sum_{j=1}^{2k-2} \rho_\mu(\check{Y}_j, \check{Y}_j).
\end{aligned} \tag{44}$$

Using (18) in (44), we get

$$\kappa(\mu) = 2 \frac{C_0}{A_0} + 4 \frac{C_1}{A_1} + 2(k-1) \left[\frac{C_3}{A_3} + \frac{C_4}{A_4} \right]. \tag{45}$$

Using the relations between the functions $A_i(\mu)$ and $C_i(\mu)$ with $A(\mu)$ and $C(\mu)$ respectively, as in (12), we obtain that the scalar curvature of a G invariant Kähler metric γ on $\mathcal{H}_{1,k}(S^2)$ has the formula (43). □

5 Einstein-Hilbert Action of $\mathcal{H}_{1,k}(S^2)$

The Einstein-Hilbert action of a Riemannian manifold (M, g) is defined by the integral

$$H(M, g) = \int_M \kappa \operatorname{vol}_g, \quad (46)$$

where κ and vol_g are the scalar curvature and the volume form respectively with respect to the Riemannian metric g on M .

Theorem 1. *The Einstein-Hilbert action of $\mathcal{H}_{1,k}(S^2)$ with respect to the L^2 metric γ_{L^2} is*

$$H(\mathcal{H}_{1,k}(S^2), \gamma_{L^2}) = \frac{2^{2k+2} \pi^{2k+1} (k+1) B_{L^2}^{2k}}{(2k)!}, \quad \forall k \geq 2. \quad (47)$$

Proof: The Einstein-Hilbert action of $\mathcal{H}_{1,k}(S^2)$ with respect to any G invariant Kähler metric γ is

$$\begin{aligned} H(\mathcal{H}_{1,k}(S^2), \gamma) &= \int_{\mathcal{H}_{1,k}(S^2)} \kappa(\mu) V(\mu) d\mu \wedge \operatorname{vol}_{G/K}, \\ &= \operatorname{Vol}(G/K) \int_1^\infty \kappa(\mu) V(\mu) d\mu, \end{aligned} \quad (48)$$

The scalar curvature of $(\mathcal{H}_{1,k}(S^2), \gamma)$, given in (43), can be written as

$$\kappa(\mu) = \frac{2}{AA'(\mu)} [2CA'(\mu) + AC'(\mu)] + (k-1) \left(B^2 - \frac{A^2}{4} \right)^{-1} [4(k+1)B - AC], \quad (49)$$

and then, by (16), we have

$$\begin{aligned} \kappa(\mu) V(\mu) &= \frac{2}{\sqrt{2}} [2ACA'(\mu) + A^2C'(\mu)] \left(B^2 - \frac{A^2}{4} \right)^{k-1} \\ &\quad + \frac{(k-1)}{\sqrt{2}} A^2 A'(\mu) [4(k+1)B - AC] \left(B^2 - \frac{A^2}{4} \right)^{k-2}, \\ &= \frac{2}{\sqrt{2}} \left(B^2 - \frac{A^2}{4} \right)^{k-1} \frac{d}{d\mu} (A^2 C) - \frac{(k-1)}{\sqrt{2}} CA^3 A'(\mu) \left(B^2 - \frac{A^2}{4} \right)^{k-2} \\ &\quad + \frac{4(k^2-1)B}{\sqrt{2}} A^2 A'(\mu) \left(B^2 - \frac{A^2}{4} \right)^{k-2}. \end{aligned} \quad (50)$$

Since

$$\frac{d}{d\mu} \left[\left(B^2 - \frac{A^2}{4} \right)^{k-1} \right] = -\frac{(k-1)}{2} A A'(\mu) \left(B^2 - \frac{A^2}{4} \right)^{k-2}. \quad (51)$$

Then,

$$\kappa(\mu) V(\mu) = \frac{2}{\sqrt{2}} \frac{d}{d\mu} \left[A^2 C \left(B^2 - \frac{A^2}{4} \right)^{k-1} \right] + 2\sqrt{2}(k^2-1)BA^2 A'(\mu) \left(B^2 - \frac{A^2}{4} \right)^{k-2}. \quad (52)$$

Hence, the Einstein-Hilbert Action $H(\mathcal{H}_{1,k}(S^2), \gamma)$ is

$$\begin{aligned} H(\mathcal{H}_{1,k}(S^2), \gamma) &= \frac{2}{\sqrt{2}} \text{Vol}(G/K) \left[A^2 C \left(B^2 - \frac{A^2}{4} \right)^{k-1} \right]_1^\infty \\ &\quad + 2\sqrt{2}(k^2-1)B^{2k-3} \text{Vol}(G/K) \int_{A(1)}^{A(\infty)} A^2 \left(1 - \frac{A^2}{4B} \right)^{k-2} dA. \end{aligned} \quad (53)$$

For the L^2 metric on $\mathcal{H}_{1,k}(S^2)$, the following limits follow from (13),

$$\begin{aligned} \lim_{\mu \rightarrow 1} A_{L^2}(\mu) &= 0, & \lim_{\mu \rightarrow \infty} A_{L^2}(\mu) &= 2B_{L^2}, \\ \lim_{\mu \rightarrow 1} C_{L^2}(\mu) &= 0, & \lim_{\mu \rightarrow \infty} C_{L^2}(\mu) &= 4(k+1), \end{aligned} \quad (54)$$

and so,

$$\lim_{\mu \rightarrow 1} \left[A_{L^2}^2 C_{L^2} \left(B_{L^2}^2 - \frac{A_{L^2}^2}{4} \right)^{k-1} \right] = \lim_{\mu \rightarrow \infty} \left[A_{L^2}^2 C_{L^2} \left(B_{L^2}^2 - \frac{A_{L^2}^2}{4} \right)^{k-1} \right] = 0. \quad (55)$$

Thus, the Einstein-Hilbert Action with respect to the L^2 metric γ_{L^2} on $\mathcal{H}_{1,k}(S^2)$ is

$$H(\mathcal{H}_{1,k}(S^2), \gamma_{L^2}) = 2\sqrt{2} (k^2-1)B_{L^2}^{2k-3} \text{Vol}(G/K) \int_{A_{L^2}(1)}^{A_{L^2}(\infty)} A_{L^2}^2 \left(1 - \frac{A_{L^2}^2}{4B_{L^2}} \right)^{k-2} dA_{L^2}. \quad (56)$$

To compute the integral above, let $t = A_{L^2}/2B_{L^2}$, then

$$H(\mathcal{H}_{1,k}(S^2), \gamma_{L^2}) = 2^4 \sqrt{2} (k^2-1)B_{L^2}^{2k} \text{Vol}(G/K) \int_0^1 t^2 (1-t^2)^{k-2} dt. \quad (57)$$

The integral in (57) is finite for all $k \geq 2$. In fact

$$\int_0^1 t^2 [1-t^2]^{k-2} dt = \frac{2^{2k-2}(k-2)! k!}{(2k)!}, \quad \forall k \geq 2. \quad (58)$$

The volume of G/K can be extracted from the formula of $\text{Vol}(\mathcal{H}_{1,k}(S^2), \gamma)$ in (17), that is,

$$\text{Vol}(G/K) = \frac{1}{\sqrt{2}} \frac{\pi^{2k+1}}{(k-1)! k!}. \quad (59)$$

Substituting (58) and (59) in (57), we get

$$H(\mathcal{H}_{1,k}(S^2), \gamma_{L^2}) = \frac{2^{2k+2} \pi^{2k+1} (k+1) B_{L^2}^{2k}}{(2k)!}. \quad (60)$$

□

By taking the holomorphic sectional curvatures $c_1 = c_2 = 4$, then the constant $B_{L^2} = \pi/2$, and so the Einstein-Hilbert action of $\mathcal{H}_{1,k}(S^2)$ with respect to the L^2 metric is

$$H(\mathcal{H}_{1,k}(S^2), \gamma_{L^2}) = \frac{2^2 \pi^{4k+1} (k+1)}{(2k)!}, \quad (61)$$

which confirms Baptista's conjectured formula (4).

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Appendix

The orthogonal complement \mathfrak{p} of the Lie algebra \mathfrak{k} in \mathfrak{g} decomposes into the $Ad(K)$ invariant subspaces [8]

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_\mu \oplus \tilde{\mathfrak{p}}_\mu \oplus \hat{\mathfrak{p}} \oplus \check{\mathfrak{p}}, \quad (62)$$

where

$$\mathfrak{p}_0 = \{(\lambda \text{diag}(i, -i, 0, \dots, 0, \text{diag}(-i, i)) : \lambda \in \mathbb{R}) \equiv \mathbb{R}, \quad (63)$$

$$\mathfrak{p}_\mu = \left\{ \left(\begin{pmatrix} 0 & x & 0 & \dots \\ -\bar{x} & 0 & 0 & \dots \\ 0 & 0 & & \\ \vdots & \vdots & & \end{pmatrix}, \begin{pmatrix} 0 & \mu x \\ -\mu \bar{x} & 0 \end{pmatrix} \right) : x \in \mathbb{C} \right\} \equiv \mathbb{C}, \quad (64)$$

$$\tilde{\mathfrak{p}}_\mu = \left\{ \left(\begin{pmatrix} 0 & -\mu \bar{y} & 0 & \dots \\ \mu y & 0 & 0 & \dots \\ 0 & 0 & & \\ \vdots & \vdots & & \end{pmatrix}, \begin{pmatrix} 0 & -\bar{y} \\ y & 0 \end{pmatrix} \right) : y \in \mathbb{C} \right\} \equiv \mathbb{C}, \quad (65)$$

$$\hat{\mathfrak{p}} = \left\{ \left(\begin{pmatrix} 0 & 0 & -\mathbf{u}^\dagger \\ 0 & 0 & \dots \\ \mathbf{u} & \vdots & \end{pmatrix}, \mathbf{0} \right) : \mathbf{u} \in \mathbb{C}^{k-1} \right\} \equiv \mathbb{C}^{k-1}, \quad (66)$$

$$\check{\mathfrak{p}} = \left\{ \left(\begin{pmatrix} 0 & 0 & \dots \\ 0 & 0 & -\mathbf{v}^\dagger \\ \vdots & \mathbf{v} & \end{pmatrix}, \mathbf{0} \right) : \mathbf{v} \in \mathbb{C}^{k-1} \right\} \equiv \mathbb{C}^{k-1}. \quad (67)$$

The almost complex structure J acts on \mathfrak{p} as

$$J : (\lambda, x, y, \mathbf{u}, \mathbf{v}) \mapsto 4\mu\lambda \frac{\partial}{\partial \mu} + (0, ix, iy, i\mathbf{u}, i\mathbf{v}). \quad (68)$$

An orthonormal basis for \mathfrak{p} with respect to the inner product $\langle, \rangle_{\mathfrak{p}}$, defined by (8), is given as follows

$$\begin{aligned} Y_0 &= \frac{i}{\sqrt{2}} (E_{11} - E_{22}, -e_{11} + e_{22}), \\ Y_1 &= (E_{12} - E_{21}, \mathbf{0}), & Y_2 &= i(E_{12} + E_{21}, \mathbf{0}), \\ Y_3 &= (\mathbf{0}, -e_{12} + e_{21}), & Y_4 &= i(\mathbf{0}, e_{12} + e_{21}), \\ \hat{Y}_{2i-1} &= (-E_{1,i+2} + E_{i+2,1}, \mathbf{0}), & \hat{Y}_{2i} &= i(E_{1,i+2} + E_{i+2,1}, \mathbf{0}), \quad i = 1, \dots, k-1 \\ \check{Y}_{2i-1} &= (-E_{2,i+2} + E_{i+2,2}, \mathbf{0}), & \check{Y}_{2i} &= i(E_{2,i+2} + E_{i+2,2}, \mathbf{0}), \quad i = 1, \dots, k-1, \end{aligned} \quad (69)$$

where $E_{\alpha\beta}$ and $e_{\alpha\beta}$ denote $(k+1) \times (k+1)$ and 2×2 matrices respectively whose element (α, β) is 1, and the others being zero.

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